SOLUTION FOR THE SEVEN CHORDS PROBLEM

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Problem Statement A regular 7-gon inscribed in a circle (ABCDEFG), with some other point H on the circle, in the arc CD. Show that the length sum of the blue chords equals the length sum of the green chords.

Solution Suppose the circumcircle has diameter 1. There is an angle β, with $-\pi/2 < \beta < -\pi/2 + \pi/7$ such that the sum of the green chords minus the sum of the blue chords is the alternating sum

$$\cos(\beta) - \cos(\beta + \pi/7) + \cos(\beta + 2\pi/7) - \cdots + \cos(\beta + 6\pi/7),$$

where all the cosine values are positive, because all the angles are in the open interval $(-\pi/2, \pi/2)$. Now replace each of the minus signs using $-\cos(x) = \cos(x + \pi)$, obtaining:

$$\cos(\beta) + \cos(\beta + \pi/7 + \pi) + \cos(\beta + 2\pi/7) + \cos(\beta + 3\pi/7 + \pi) + \cos(\beta + 4\pi/7) + \cos(\beta + 5\pi/7 + \pi) + \cos(\beta + 6\pi/7)$$
$$= \cos(\beta) + \cos(\beta + 8\pi/7) + \cos(\beta + 2\pi/7) + \cos(\beta + 10\pi/7) + \cos(\beta + 4\pi/7) + \cos(\beta + 12\pi/7) + \cos(\beta + 6\pi/7).$$

Rearrange to get the sum:

$$\sum_{k=0}^{6} \cos\left(\beta + k(2\pi/7)\right).$$

This sum is the real part of a geometric sequence of complex numbers adding to 0:

$$\sum_{k=0}^{6} e^{i\beta} e^{ki\frac{2\pi}{7}}$$
$$= e^{i\beta} \frac{1 - e^{7i\frac{2\pi}{7}}}{1 - e^{i\frac{2\pi}{7}}}$$
$$= e^{i\beta} \frac{1 - e^{2\pi i}}{1 - e^{i\frac{2\pi}{7}}} = 0$$

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