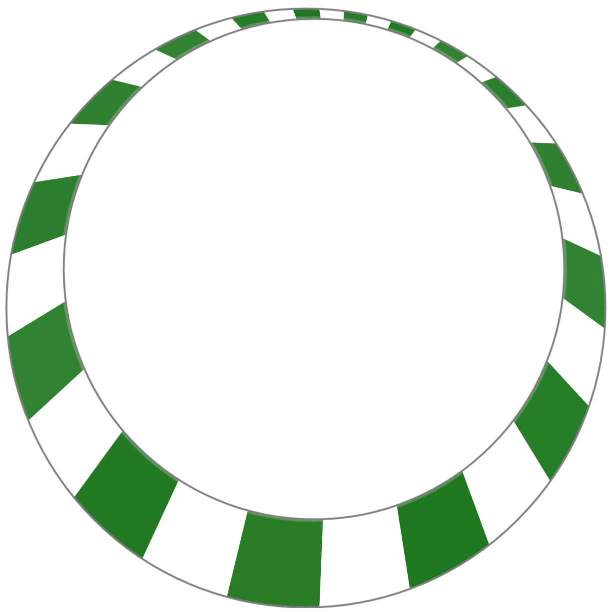
THE CHESHIRE CAT GRINS

MARSHALL BUCK



Problem. What is the total area of the green teeth relative to the large circle area? (Assume the inner circle radius is 80% of the outer. The 32 lines forming the tooth sides are concurrent, and the lines are spaced at equal angles apart from one another.)

Note. It will turn out that the number of teeth can be any even number, and the answer will be the same. Also, the answer is independent of the positioning of the two circles relative to the spray of rays from the concurrency point. Although I have a proof for any even number, the proof I will give here assumes that the number of teeth is divisible by 4.

First recall

Theorem 1 (Proposition 11 of Archimedes's Book of Lemmas)**.** *If two chords AB, CD in a circle intersect at right angles in a point O, not being the center, then*

$$AO^2 + BO^2 + CO^2 + DO^2 = (diameter)^2.$$

Proof. Suppose the circle has radius r and is centered at $(0,0)$. Rotate the circle so that AB becomes parallel to the x-axis and CD is parallel to the y-axis, and $O = (x, y)$ is the intersection of the chords. Let $a = |OA|, b = |OB|, c = |OC|, d = |OD|$. Then

$$a = \sqrt{1-y^2} + x$$
$$b = \sqrt{1-y^2} - x$$
$$c = \sqrt{1-x^2} - y$$
$$d = \sqrt{1-x^2} + y$$

and

$$\begin{aligned} a^2 + b^2 &= 2(1-y^2) + 2x^2 \\ c^2 + c^2 &= 2(1-x^2) + 2y^2 \\ a^2 + b^2 + c^2 + d^2 &= 4. \end{aligned}$$

$\square$

Divide the 32 tooth-side rays from the concurrency point O (necessarily inside both circles) into two interleaved sets A and B of size 16 each. Let the rays in A be the "right sides" of the teeth (at least in the bottom of the grin, but then continued around consistently). Then the rays of B will be the "left sides" of the teeth. A ray from set A when rotated clockwise about O to the next B ray, sweeps out a wedge containing one of the green teeth at its outer part. By decomposing a sweep into a sequence of tiny angles, one sees that the area swept out together by a ray, its opposite ray and the two perpendicular rays all turning the angular distance θ, will be $2r^2\theta$, since by the theorem above, the sum of the squared ray lengths to a circle of radius r will be $4r^2$. If the outer circle has radius r and the inner circle has radius s, then the difference between the sweep using the outer circle and the sweep using the inner circle will be $2(r^2 - s^2)\theta$. For our Cheshire cat, $\theta = 2\pi/32 = \pi/16$, and such a sweep would make 4 teeth. The total 16 teeth would be 4 times larger: $4 \cdot 2(r^2 - s^2)\pi/16 = \pi(r^2 - s^2)/2$, and the full circle has area πr^2, so the tooth coverage ratio is $(1-(s/r)^2)/2$.