THE THREE-BODY PROBLEM

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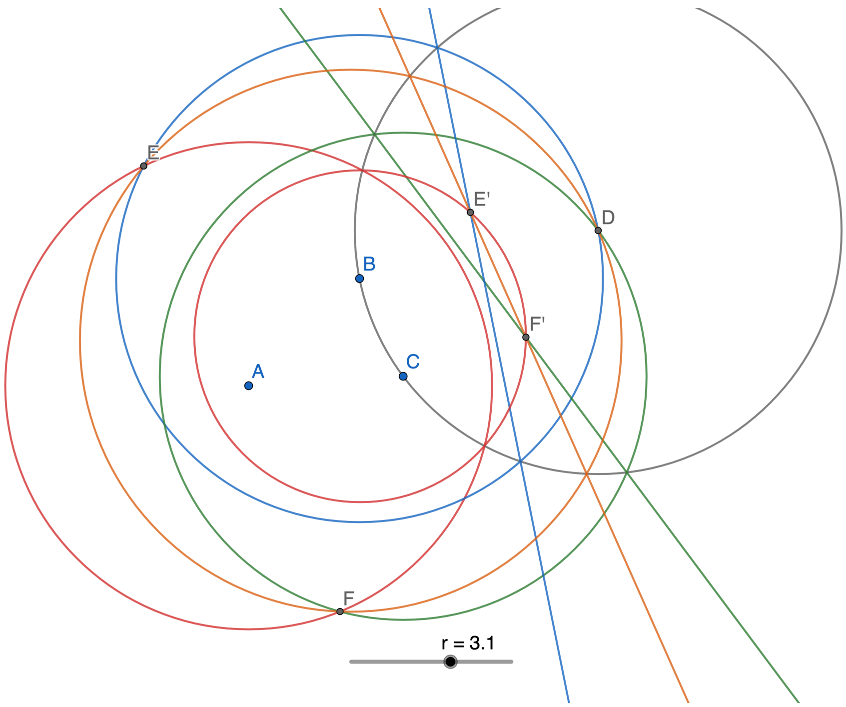

Theorem 1. *Three unit disks A, B, C intersect so that none of the individual disks covers all three "intersection petals" $A \cap B$, $B \cap C$, $C \cap A$. Then there is a disk of radius at most $\frac{2}{\sqrt{3}}$ that does cover all three petals simultaneously. In fact, one such disk is the disk bounded by the circumcircle of the external endpoints of the petals.*

Proof. The petal ends form a triangle $\triangle DEF$. One of the angles must be at most 60°. Say that is the point D. Invert the diagram with respect to the unit circle about D, which is the black circle going through the ponts B, C. The blue and green disk boundaries turn into blue and green lines L, M (not labeled in the diagram) at distance $1/2$ from D. The red disk boundary inverts into a red circle containing the intersection of the two lines: $L \cap M$. The other two petal ends, E, F, invert into points E' on L and F' on M. Furthermore, we still have $\angle E'DF' \leq 60°$. Then one can show (using the next Lemma) that the distance from the orange line $\overline{E'F'}$ to D is at least $\frac{\sqrt{3}}{4}$,

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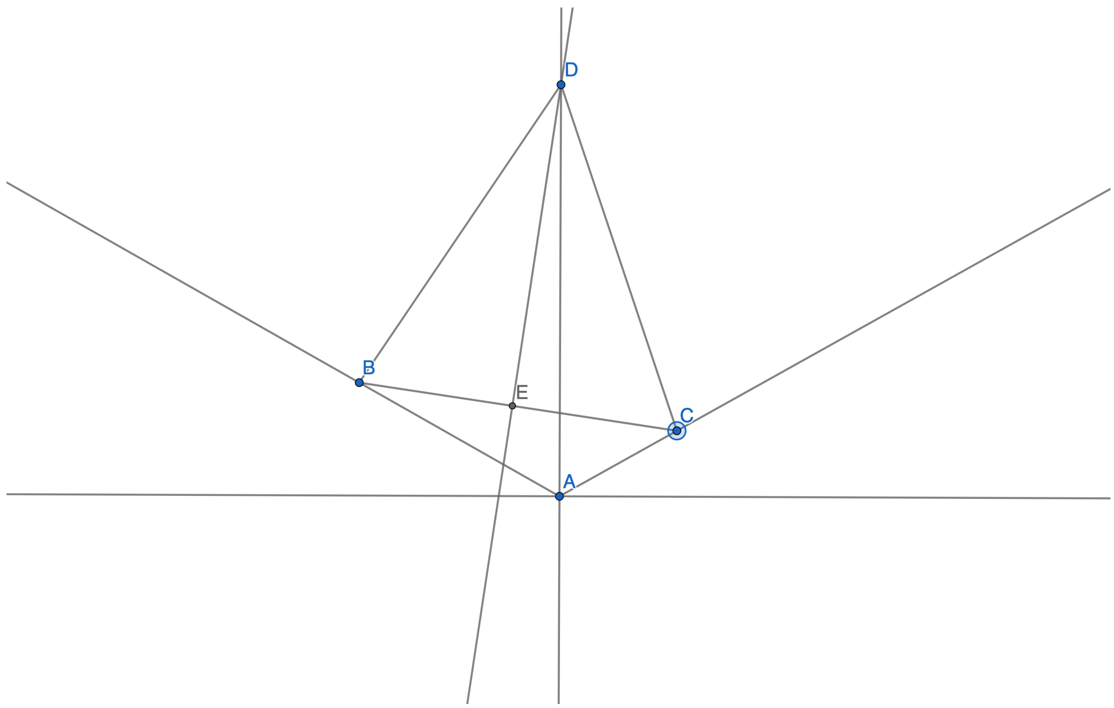
so that the diameter of the orange circumcircle (DEF) (the inverse image of the line $\overline{E'F'}$) is at most $\frac{4}{\sqrt{3}}$. We also know that circumcircle contains all the petals, because in the inverse diagram all the petal images are on the same side of the orange line $\overline{E'F'}$. □

The drawing is not quite right, since I insist the angle at D ($\angle EDF$) is at most 60°.

Lemma 1. *Say that the rays $\overline{AB}$ and $\overline{AC}$ are at distance 1 from D and $\angle BDC \leq 60°$. Then the distance from D to $\overline{BC}$ is at least $\frac{\sqrt{3}}{2}$.*



Proof. Points on the line $\overline{BC}$ that are not on the segment $\overline{BC}$ are below the $\overline{AB}$ or $\overline{AC}$ rays, so are at distance more than 1. Thus, we just need to show that the points on the segment $\overline{BC}$ all have distance at least $\frac{\sqrt{3}}{2}$ from D. This is true because $|DB| \geq 1$, $|CD| \geq 1$, and $\angle BDC \leq 60°$. The distance is minimized when $|DB| = 1$, $|CD| = 1$, and $\angle BDC = 60°$, when the distance is $\frac{\sqrt{3}}{2}$. □